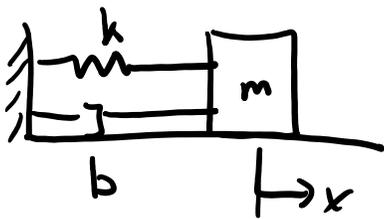


- Office hours tomorrow moved to 4:10-5:00 PM
- No class Monday (MLK day)
- HW #2 due Tuesday

Second Order Mass-Spring-Damper



mass on surface m

linear spring with stiffness k

linear damper with damping coeff b

$$m, k, b \geq 0$$

$$m\ddot{x} + b\dot{x} + kx = 0$$

second order homogenous constant coefficient ordinary differential equation implies linear

$$x(t) = \Sigma e^{st}, \quad \dot{x}(t) = s \Sigma e^{st}, \quad \ddot{x}(t) = s^2 \Sigma e^{st}$$

$$ms^2 \Sigma e^{st} + bs \Sigma e^{st} + k \Sigma e^{st} = 0$$

$$(ms^2 + bs + k) \Sigma e^{st} = 0$$

$$ms^2 + bs + k = 0$$

solve for s

$$s^2 + \frac{b}{m}s + \frac{k}{m} = 0$$

$$s = \frac{-\frac{b}{m} \pm \sqrt{\frac{b^2}{m^2} - 4\frac{k}{m}}}{2} = -\frac{b}{2m} \pm \frac{1}{2} \sqrt{\frac{b^2}{m^2} - 4\frac{k}{m}}$$

Scenario #1 $b=0$ "undamped"

$$s = \pm \frac{1}{2} \sqrt{-4 \frac{k}{m}} = \pm \sqrt{\frac{k}{m}} i$$

$$i = \sqrt{-1}$$

$$x(t) = \sum e^{\pm \sqrt{\frac{k}{m}} i t} \Rightarrow x(t) = A \sin(\sqrt{\frac{k}{m}} t + \phi)$$

\downarrow amplitude \swarrow phase angle
 \uparrow
 A, ϕ calculated from initial conditions

Euler's relation
 $e^{i\theta} = \cos\theta + i \sin\theta$

$$\omega_n \triangleq \sqrt{\frac{k}{m}} \quad \text{"natural frequency"}$$

Scenario #2 $b > 0$, $\frac{b^2}{m^2} = 4 \frac{k}{m}$ "critically damped"

$$\zeta = 1$$

$$x(t) = \sum e^{-\frac{b}{2m} t}$$

$$\frac{b^2}{m^2} = 4 \omega_n^2 \Rightarrow b = \sqrt{4 m^2 \omega_n^2} = 2 m \omega_n$$

$$b_r \triangleq 2 m \omega_n \quad \text{"critical damping coefficient"}$$

Scenario #3 $b > 0$, $\frac{b^2}{m^2} > 4 \frac{k}{m}$, $\zeta > 1$

$$b > 2 m \omega_n = b_r$$

$$\zeta \triangleq \frac{b}{b_r} = \frac{b}{2 m \omega_n}$$

$$\zeta > 1$$

\hookrightarrow damping ratio

$$s = -\frac{b}{2m} \pm \frac{1}{2} \sqrt{\underbrace{\frac{b^2}{m^2} - 4 \omega_n^2}_{> 0}}$$

$$b = \zeta 2 m \omega_n$$

$$s = -\zeta \omega_n \pm \frac{1}{2} \sqrt{\zeta^2 4 \omega_n^2 - 4 \omega_n^2} = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$$

$$X(t) = \sum e^{(\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1})t} \quad \text{exponential decay}$$

"over damped"

Scenario #4 $b > 0, \frac{b^2}{m^2} < 4\frac{k}{m}, \zeta < 1$

$$b < b_r = 2m\omega_n$$

$$s = -\zeta\omega_n \pm \omega_n\sqrt{(\zeta^2-1)} = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}i$$

$$X(t) = \sum e^{(-\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}i)t}$$

↓ rewrite with Euler's, etc

$$X(t) = A e^{-\zeta\omega_n t} \sin(\omega_n\sqrt{1-\zeta^2}t + \phi)$$

$\omega_d \triangleq \omega_n\sqrt{1-\zeta^2}$ "damped natural frequency"

$$X(t) = A e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$$

"under damped" $\zeta < 1$