

Method's for arriving at the Equations of Motion

- E.O.M
- (I) Direct method (Newton-Euler Method)
 - draw a free body diagram for each element and formulate $\sum \vec{F} = \frac{d\vec{p}}{dt}$ and $\sum \vec{T} = \frac{d\vec{h}}{dt}$.
 - (II) Indirect method (Lagrange's method)
 - draw free body diagram of entire system, formulate the kinetic & potential energy of the system and then Lagrange's Equation.

Lagrange's Method
Energy-based method

Joseph-Louis Lagrange 1736-1813
Mécanique Analytique 1788

Kinetic Energy of a translating and rotating planar

rigid body, T

$$T = \underbrace{\frac{1}{2} m v_c^2}_{\text{translation}} + \underbrace{\frac{1}{2} I_c \omega^2}_{\text{rotation}}$$

↑ Kinetic energy of a single body
 ↑ total mass of the RB
 ↑ magnitude of the velocity of mass center
 ↑ Moment of inertia about the mass center
 ↑ angular velocity of the body

For a system of rigid bodies we sum the energy:

$$T_{\text{sys}} = \frac{1}{2} \sum_{i=1}^N m_i v_{ci}^2 + \frac{1}{2} \sum_{i=1}^N I_{ci} \omega_i^2 \quad \text{for } N \text{ rigid bodies or particles}$$

$\underbrace{\hspace{10em}}$
 neglect for any particles

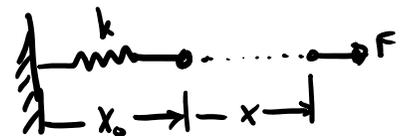
Potential Energy, U

describe conservative forces affecting the system, those that store and release energy with no losses (that are associated with changes in the generalized coordinates)

mass in a constant gravity field



potential energy in a spring



$$F = kx$$

$$P = F \dot{x} = kx \frac{dx}{dt}$$

$$U = \int P dt = \int kx \frac{dx}{dt} dt$$

$$U = \frac{kx^2}{2}$$

For a system sum all potential energy contributions:

$$U_{\text{sys}} = \sum_{i=1}^M U_i$$

M # of conservative forces applied to the system

Lagrangian Equations

If x is a generalized coordinate and v is generalized speed such that $v = \dot{x}$ and the kinetic energy is $T = \frac{1}{2} m v^2$

note this fact:

$$\frac{\partial T}{\partial v} = \frac{\partial}{\partial v} \left(\frac{1}{2} m v^2 \right) = m v = p \rightarrow \text{momentum}$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial v} \right) = \frac{dp}{dt} \quad \text{if } m \text{ is constant}$$
$$\frac{d}{dt} \left(\frac{\partial T}{\partial v} \right) = m \frac{dv}{dt}$$

Similar for potential energy:

assume $U = mgx$



$$-mg = m \frac{dv}{dt}$$

$$\frac{\partial U}{\partial x} = mg$$

So for systems with conservative forces we can see that Newton's 2nd Law can be written as:

$$\frac{d}{dt} \left(\frac{\partial T}{\partial v} \right) = - \frac{\partial U}{\partial x}$$

Lagrange's of the second kind for single particle.

For a conservative unconstrained system with n DoFs
(motion constraints)

with GC's x_i and GS's v_i for $i=1, \dots, n$ we can write Lagrange's Equations of the second kind.

$$L_{\text{sys}} \triangleq T_{\text{sys}} - U_{\text{sys}} : \text{Lagrangian}$$

$$\frac{d}{dt} \left(\frac{\partial L_{\text{sys}}}{\partial v_i} \right) = \frac{\partial L_{\text{sys}}}{\partial x_i} \quad \text{for } i=1, \dots, n$$

These are the equations of motion!

If there are non-conservative forces (causes energy loss)
then use Lagrange's Equation of the first kind:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial v_i} \right) = \frac{\partial L}{\partial x_i} + Q_i \quad \text{for } i=1, \dots, n$$

$Q_i \triangleq$ generalized force

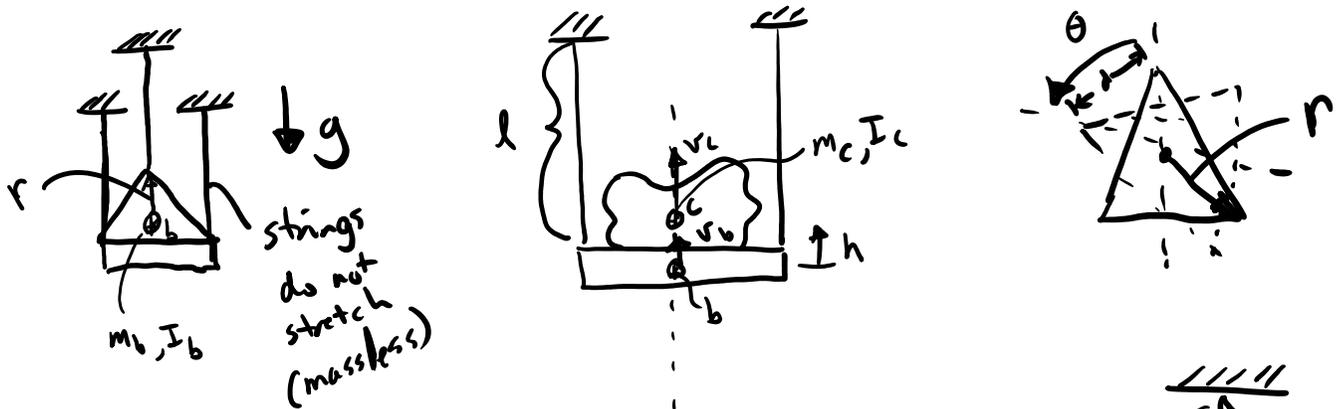
These describe the contribution of a non-conservative force
to changes in the i th G.C. and G.S.

Steps in modeling and analysis using Lagrange's equations

- ① Define a specific question to answer with the model. This is critical for making the simplest model (making assumptions) to answer the question.
- ② Identify important elements/components in the system (rigid bodies, particles, forces, energy storage, etc.)
- ③ Draw a free body diagram (FBD) of the system
- ④ Define the generalized coordinates on the FBD as well as any forces/torques.
- ⑤ Write expressions for v and ω for each particle/body in terms of the G.C.'s and G.S.'s
- ⑥ Write expressions for T , U , Q , and L .
- ⑦ Take derivatives and form Lagrange's equations.
- ⑧ Linearize the equations of motion if desired.
- ⑨ Seek solutions to the ODEs (analytically or numerically) to get trajectories of the coordinates and speeds.
- ⑩ Interpret the results.

Trifilar Pendulum

Q: How does the object's moment of inertia relate to the period of oscillation?



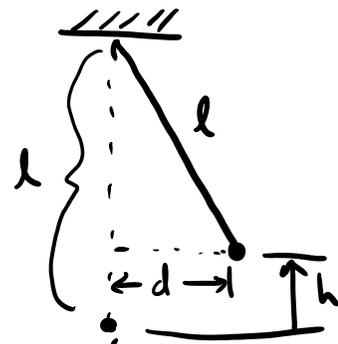
$$v_b = v_c = v$$

ω angular velocity of objects about vertical axis



$$\frac{d}{2} = r \sin\left(\frac{\theta}{2}\right)$$

$$d = 2r \sin\left(\frac{\theta}{2}\right)$$



$$l^2 = d^2 + (l-h)^2$$

$$h = l - \sqrt{l^2 - d^2}$$

$$h = l - \sqrt{l^2 - (2r \sin \frac{\theta}{2})^2} \quad \theta(t)$$

$$v = \frac{dh}{dt}$$

$$T = \frac{1}{2} m_b v^2 + \frac{1}{2} I_b \omega^2 + \frac{1}{2} m_c v^2 + \frac{1}{2} I_c \omega^2$$

$$U = (m_b + m_c) g h = (m_b + m_c) g \left(l - \sqrt{l^2 - (2r \sin \frac{\theta}{2})^2} \right)$$

$$L = T - U$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = \frac{\partial L}{\partial \theta}$$

$$\dot{\theta} = \omega$$

1 equation of motion for SDoF