

— canonical form

$$M \ddot{\bar{s}} + C \dot{\bar{s}} + K \bar{c} = 0$$

MDOF linear system, no forcing

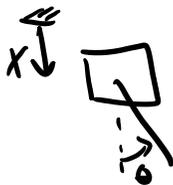
$\bar{c}$ : generalized coordinate vector  
 $\dot{\bar{s}}$ : " speed vector

$n$  DoF, then  $\bar{c}: n \times 1$   
 $\dot{\bar{s}}: n \times 1$   
 $M, C, K: n \times n$  matrices

$M$ , mass matrix, will be symmetric and positive definite.

$M$  is positive definite if  $\bar{x}^T M \bar{x} > 0$  for any  $\bar{x}$ .

$C$  and  $K$  may not symmetric or positive definite.

Bouncy Pendulum 

$$M = \begin{bmatrix} I_c + ml^2 & 0 \\ 0 & M + m \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \quad K = \begin{bmatrix} Mgl & 0 \\ 0 & k \end{bmatrix}$$

$$\bar{c} = \begin{bmatrix} \theta \\ x \end{bmatrix} \quad \dot{\bar{s}} = \begin{bmatrix} \omega \\ v \end{bmatrix}$$

- linear equations that are uncoupled  $\Rightarrow$  two separate ODEs

$$(I_c + ml^2) \dot{\omega} + Mgl \theta = 0$$

$$\omega_{n\theta}^2 = \frac{Mgl}{I_c + ml^2}$$

$$(M + m) \dot{v} + bv + kx = 0$$

$$\omega_{nx}^2 = \frac{k}{M + m}$$

Ball-Channel Pendulum





$$M = \begin{bmatrix} m_p l^2 + m_b r^2 & -m_b r^2 \\ -m_b r^2 & m_b r^2 \end{bmatrix} \quad \bar{c} = \begin{bmatrix} \theta \\ \phi \end{bmatrix} \quad \bar{s} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad K = \begin{bmatrix} g l m_p & g m_b r \\ g m_b r & g m_b r \end{bmatrix}$$

$$(m_p l^2 + m_b r^2) \ddot{\theta} - m_b r^2 \ddot{\phi} + g l m_p \theta + g m_b r \phi = 0$$

$$-m_b r^2 \ddot{\theta} + m_b r^2 \ddot{\phi} + g m_b r \theta + g m_b r \phi = 0$$

- coupled linear equations  $\Rightarrow$  must solve simultaneously

-  $M$  &  $K$  symmetric

Start with the no-damping case  $C=0$ :

$$M \ddot{\bar{c}} + K \bar{c} = 0$$

$\underbrace{\hspace{10em}}_{L \text{ both symmetric}}$

Define the Cholesky decomposition of  $M$ :

$$M = L L^T$$

where  $L$  is the lower triangular decomposition of  $M$

Now define the transform of coordinates:

$$\bar{q} = L^T \bar{c}$$

$\bar{q}$ : new coordinates that are related the G.C.S. "mass"

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$\bar{q}$ : new coordinates that are related to the G.C.S., "mass normalized coordinates"

$$M (L^T)^{-1} \ddot{\bar{q}} + K (L^T)^{-1} \bar{q} = 0$$

$$L^{-1} M (L^T)^{-1} \ddot{\bar{q}} + L^{-1} K (L^T)^{-1} \bar{q} = 0$$

$$L^{-1} L L^T (L^T)^{-1} = I \text{ identity}$$

$$\boxed{I \ddot{\bar{q}} + \tilde{K} \bar{q} = 0}$$

mass normalized equations equivalent to  $\ddot{c} + \frac{K}{m} c = 0$

$$\tilde{K} = L^{-1} K (L^T)^{-1}$$

↳ guaranteed to be symmetric if  $K$  is symmetric

What is the solution?

Assume:  $\bar{q}(t) = \bar{q}_0 \sin \omega t$

$$\dot{\bar{q}}(t) = \bar{q}_0 \omega \cos \omega t$$

$$\ddot{\bar{q}}(t) = -\bar{q}_0 \omega^2 \sin \omega t$$

$$I (-\bar{q}_0 \omega^2 \sin \omega t) + \tilde{K} (\bar{q}_0 \sin \omega t) = 0$$

$$\boxed{(\tilde{K} - \omega^2 I) \bar{q}_0 \sin \omega t = 0}$$

algebraic equation

$\bar{q}_0 = 0$  trivial solution

$\sin \omega t \neq 0$  for all  $t$

$$\tilde{K} - \omega^2 I \neq 0$$

similar -

$$\tilde{K} - \omega^2 I \neq 0$$

$(\tilde{K} - \omega^2 I) \bar{q}_0 = 0 \Rightarrow$  non-trivial solution from this equation

$$\tilde{K} \bar{q}_0 = \omega^2 \bar{q}_0$$

This is a classic eigenvalue problem and since  $\tilde{K}$  is symmetric it is a "symmetric eigenvalue problem".

$$A \bar{x} = \lambda \bar{x}$$

$\lambda_j$ : eigenvalues  
 $\bar{x}_j$ : eigenvectors

A is nxn then we get n eigenvalues and eigenvectors

if A is symmetric:  
eigenvalues are real  
eigenvectors are real

if A is positive definite:  
eigen values are positive

Recall a unique solution for  $\lambda$  can be found with:

$$\det(A - \lambda I) = 0$$

$$A \bar{x} - \lambda \bar{x} = 0 \\ (A - \lambda I) \bar{x} = 0$$

Thus:

$$\det(\tilde{K} - \omega^2 I) = 0$$

Example: 2x2  $\tilde{K}$  matrix

$$\tilde{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}$$

$$\det \left( \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} - \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix} \right) = 0$$

$$\det \left( \begin{bmatrix} k_{11} - \omega^2 & k_{12} \\ k_{12} & k_{22} - \omega^2 \end{bmatrix} \right) = 0$$

$$(k_{11} - \omega^2)(k_{22} - \omega^2) - k_{12}^2 = 0$$

$$\omega^4 - \omega^2 k_{22} - \omega^2 k_{11} - k_{11} k_{22} - k_{12}^2 = 0 \quad \text{quartic in } \omega$$

$$\lambda \triangleq \omega^2$$

$$\lambda^2 - (k_{11} + k_{22})\lambda - k_{11}k_{22} - k_{12}^2 = 0 \quad \text{quadratic in } \lambda$$

highest order  
of  $\lambda = n$  (# of DoF)  $\rightarrow$  characteristic equation of the system  
in this case a quadratic equation

$$\lambda_{1,2} = \frac{k_{11} + k_{22}}{2} \pm \frac{\sqrt{(k_{11} + k_{22})^2 - 4(k_{11}k_{22} - k_{12}^2)}}{2} \quad \text{real valued}$$

$\hookrightarrow$  two eigenvalues for the 2 DoF system

These two e-vals correspond "eigen frequencies"

$$\omega_{1,3} = \pm \sqrt{\lambda_1} \quad \omega_{2,4} = \pm \sqrt{\lambda_2}$$

These are the natural frequencies of the system.

$$\bar{q}(t) = b_1 \bar{q}_{01} \sin \omega_1 t + b_2 \bar{q}_{02} \sin \omega_2 t + b_3 \bar{q}_{01} \sin \omega_3 t + b_4 \bar{q}_{02} \sin \omega_4 t$$

$$\bar{q}(t) = d_1 \bar{q}_{01} \sin(\omega_1 t + \phi_1) + d_2 \bar{q}_{02} \sin(\omega_2 t + \phi_2)$$