

$$M \ddot{\bar{s}} + C \dot{\bar{s}} + K \bar{c} = 0$$

if $C=0$ and K is symmetric:

$$\ddot{\bar{q}} + \bar{K} \bar{q} = 0 \quad \bar{q} = L^T \bar{c}$$

$$M = L L^T$$

$$\hat{K} = L^{-1} K (L^T)^{-1}$$

assumed the $\bar{q}(t) = \bar{q}_0 \sin \omega t$

$$(\hat{K} - \omega^2 I) \bar{q}_0 = 0$$

solve for eigenvalues

$$\det(\hat{K} - \omega^2 I) = 0 \Rightarrow \text{characteristic equation a polynomial}$$

$$\omega^2 = \lambda$$

$$\det(\hat{K} - \lambda I) = 0$$

↑ look for λ, λ are the eigenvalues of \hat{K}

char. eq polynomial is of order n , where n is number DoF.

$$q(t) = d_1 \bar{q}_{01} \sin(\omega_1 t + \phi_1) + d_2 \bar{q}_{02} \sin(\omega_2 t + \phi_2)$$

Eigenvectors

$$(\bar{K} - \lambda I) \bar{q}_0 = 0$$

plug in λ 's to this equation and solve for \bar{q}_0

$$\begin{bmatrix} k_{11} - \lambda_1 & k_{12} \\ k_{12} & k_{22} - \lambda_1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

two equations
and two
unknowns, q_1 and q_2

$$(k_{11} - \lambda_1) q_1 + k_{12} q_2 = 0$$

$$k_{12} q_1 + (k_{22} - \lambda_1) q_2 = 0$$

$$\Rightarrow \frac{-(k_{11} - \lambda_1)(k_{22} - \lambda_1)}{k_{12}} q_2 + k_{12} q_2 = 0$$

$$q_2 = ? = 0$$

no unique solution

so pick $q_2 = 1$

$$(k_{11} - \lambda_1) q_1 + k_{12} = 0 \Rightarrow$$

$$q_1 = \frac{-k_{12}}{k_{11} - \lambda_1} \quad q_2 = 1$$

eigenvector components
associated with λ_1

$$q_1 = \frac{-k_{12}}{k_{11} - \lambda_2} \quad q_2 = 1$$

associated with λ_2

$$\bar{q}_{01} = \begin{bmatrix} \frac{-k_{12}}{k_{11} - \lambda_1} \\ 1 \end{bmatrix}$$

$$\bar{q}_{02} = \begin{bmatrix} \frac{-k_{12}}{k_{11} - \lambda_2} \\ 1 \end{bmatrix}$$

eigenvectors

eigenvectors are orthogonal since \tilde{K} is symmetric

$$\hat{q}_{01} = \frac{\bar{q}_{01}}{\|\bar{q}_{01}\|} \quad \hat{q}_{02} = \frac{\bar{q}_{02}}{\|\bar{q}_{02}\|} \quad \text{unit eigenvectors}$$

General Solution for the set ODEs:

$$\bar{q}(t) = \sum_{i=1}^n d_i \bar{q}_{0i} \sin(\omega_i t + \phi_i)$$

$$\phi_i = \arctan \left(\frac{\omega_i \hat{q}_{0i}^T \bar{q}(0)}{\hat{q}_{0i}^T \dot{\bar{q}}(0)} \right)$$

$$d_i = \frac{\hat{q}_{0i}^T \bar{q}(0)}{\sin \phi_i} \quad \omega_i = \sqrt{\lambda_i}$$

↑
modal participation factor

mode shapes

in terms of \bar{q}

Define

$$P \triangleq \begin{bmatrix} \hat{q}_{01} & \hat{q}_{02} \end{bmatrix}$$

unit eigenvectors

columns of P are orthonormal

$$P^{-1} = P^T$$

$$\bar{q} = L^T \bar{c}$$

$$S \triangleq (L^T)^{-1} P = \begin{bmatrix} \bar{u}_1 & \bar{u}_2 \end{bmatrix}$$

in terms of \bar{c}

$$S \cong (L^T) \quad \Gamma = L \begin{matrix} u_1 & u_2 \\ \vdots & \vdots \end{matrix}$$

"mode shapes" in terms of \bar{c}

$$\bar{c}(t) = \sum_{i=1}^n d_i \bar{u}_i \sin(\omega_i t + \phi_i)$$

final solution form

$$\bar{c}(t) = (L^T)^{-1} \hat{q}$$

$$\bar{u}_i = (L^T)^{-1} \hat{q}_{0i}$$

$$(L^T)^{-1} \hat{q} = (L^T)^{-1} \sum_{i=1}^n d_i \hat{q}_{0i} \sin(\omega_i t + \phi_i)$$

$$\ddot{\bar{q}} + \hat{K} \bar{q} = 0$$

define \bar{r} such that $\bar{q} = P \bar{r}$

sub in $\bar{q} = P \bar{r}$

$$P \ddot{\bar{r}} + \tilde{K} P \bar{r} = 0$$

$$P^{-1} P \ddot{\bar{r}} + P^{-1} \tilde{K} P \bar{r} = 0$$

$$\ddot{\bar{r}} + P^T \tilde{K} P \bar{r} = 0$$

define $\Delta = P^T \tilde{K} P$

$$\ddot{\bar{r}} + \Delta \bar{r} = 0$$

Δ will be diagonal and have the eigenvalues on the diagonal.

$$\underline{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\begin{bmatrix} \ddot{r}_1 \\ \ddot{r}_2 \end{bmatrix} + \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = 0$$

Decoupled differential equations!

$$\begin{aligned} \ddot{r}_1 + \lambda_1 r_1 = 0 &\Rightarrow r_1 = A_1 \sin(\omega_1 t + \phi_1) & \omega_1 = \sqrt{\lambda_1} \\ \ddot{r}_2 + \lambda_2 r_2 = 0 & r_2 = A_2 \sin(\omega_2 t + \phi_2) & \omega_2 = \sqrt{\lambda_2} \end{aligned}$$