

- HW # 8 Due Wednesday

State Space Form of Equations of Motion

Canonical form:

$$M \ddot{s} + C \dot{s} + K \bar{c} = \bar{F}$$

$$\dot{s} + M^{-1}C \bar{s} + M^{-1}K \bar{c} = M^{-1}\bar{F}$$

$$\left. \begin{aligned} \dot{s} &= M^{-1}\bar{F} - M^{-1}C \bar{s} - M^{-1}K \bar{c} \\ \dot{\bar{c}} &= \bar{s} \end{aligned} \right\}$$

explicit 1st order form of the linear ODEs

now form the state vector:

$$\bar{x} = \begin{bmatrix} \bar{c} \\ \bar{s} \end{bmatrix} \quad \dot{\bar{x}} = \begin{bmatrix} \dot{\bar{c}} \\ \dot{\bar{s}} \end{bmatrix}$$

now we can write equation in state space form:

$$\dot{\bar{x}} = A \bar{x} + B \bar{u}$$

A: state matrix

$$\bar{u} = \bar{F}$$

B: input matrix

\bar{x} : state vector

\bar{u} : input vector

$$\dot{\bar{s}} = M^{-1}\bar{F} - M^{-1}C \bar{s} - M^{-1}K \bar{c}$$

$$\dot{\bar{c}} = \bar{s}$$

$$\begin{bmatrix} \dot{\bar{c}} \\ \dot{\bar{s}} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}}_A \begin{bmatrix} \bar{c} \\ \bar{s} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}}_B \bar{F}$$

$\begin{matrix} 2n \times 2n \\ \begin{matrix} nxn & nxn \end{matrix} \\ \begin{matrix} 2n \times 1 \\ \begin{matrix} nx1 \\ nx1 \end{matrix} \end{matrix}$

Solve for the homogeneous solution, which is for

$$\dot{\bar{x}} = A \bar{x} \quad 2n \text{ equations}$$

$$\text{Assume } \bar{x}(t) = \bar{X} e^{st}, \quad \dot{\bar{x}}(t) = s \bar{X} e^{st}$$

$$s \bar{X} e^{st} = A \bar{X} e^{st}$$

$$[sI - A] \bar{X} e^{st} = 0$$

$e^{st} = 0$: can't be true for all t

$\bar{X} = 0$: trivial solution

$[sI - A] = 0$: can't be true in general

$[sI - A] \bar{X} = 0 \Rightarrow$ non-trivial solution!

$s \bar{X} = A \bar{X} \Rightarrow$ eigenvalue problem

$$\det |sI - A| = 0 \quad \text{we can solve for } \underline{s}$$

polynomial in s
polynomial of order $2n$
characteristic equation

s_1, \dots, s_{2n} are the $2n$ eigenvalues of A

$\text{eig}(A)$ gives s_1, \dots, s_{2n} in python

General form of s is:

$$s = \lambda_{1,2} \pm \omega i \quad \text{where } i = \sqrt{-1}$$

$$s_1 = \lambda + \omega i$$

$$s_2 = \lambda - \omega i$$

if all s are imaginary you get n complex conjugate pairs from A .

thus:

$$\bar{x} = e^{(\lambda + \omega i)t}$$

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thus:

$$\bar{x}(t) = \bar{X} e^{(\lambda + \omega i)t}$$

$$\bar{x}(t) = \bar{X} [e^{(\lambda + \omega i)t} + e^{(\lambda - \omega i)t}]$$

} both are solutions

If λ or ω happen to be zero, s can take these forms:

$$s_{1,2} = \lambda \pm \omega i$$

complex conj. pair with real & imaginary part

$$s_{1,2} = \lambda_1, \lambda_2$$

two real values

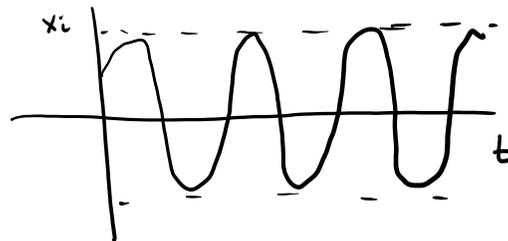
$$s_{1,2} = \pm \omega i$$

complex conjugate pair with only imaginary part

These correspond to the possible solutions:

Undamped mode: $s_{1,2} = \pm \omega i$

$$\bar{x}(t) = \bar{X} A \sin(\omega t + \phi)$$



Exponential growth or decay, no oscillation:

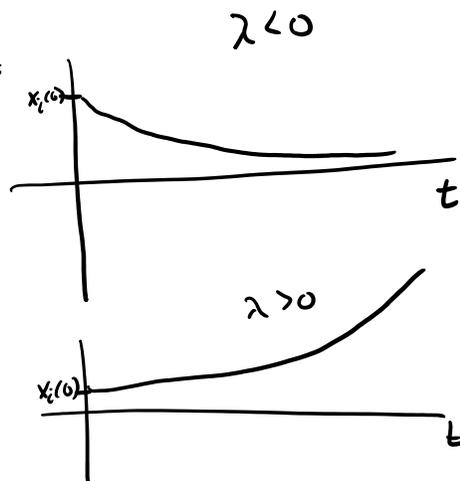
$$s_{1,2} = \lambda_1, \lambda_2$$

if decay: critically damped & over damped are in this category

$$\bar{x}(t) = \bar{X} A e^{\lambda_1 t} \quad \text{and} \quad \bar{x}(t) = \bar{X} A e^{\lambda_2 t}$$

$\lambda > 0 \Rightarrow$ growth, unstable

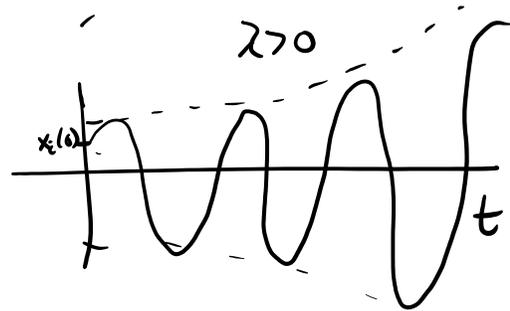
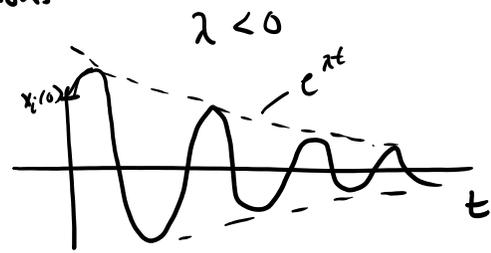
$\lambda < 0 \Rightarrow$ decay, stable



Exponential decay or growth with oscillations

$$s_{1,2} = \lambda \pm \omega i$$

$$\bar{x}(t) = \bar{X} A e^{\lambda t} \sin(\omega t + \phi)$$



$S = -2.3 \pm 6.1 i$ under damped motion

$S = 12.3$ exponential growth, no oscillation

$S = \pm 0.3 i$ undamped motion (steady oscillation)

$S = 0.2 \pm 12.0 i$ exponential growth, with oscillation

$e^{12.3t}$ vs $e^{0.2t}$

For each mode (each eigenvalue or pair of eigenvalues):

$\lambda = 0 \Rightarrow$ no decay or growth

$\omega = 0 \Rightarrow$ no oscillation

$\lambda > 0 \Rightarrow$ growth, unstable

$\omega > 0 \Rightarrow$ oscillation

$\lambda < 0 \Rightarrow$ growth, stable

$\lambda < 0 \Rightarrow$ growth, stable

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There is corresponding eigenvector for each eigenvalue (or complex pair of eigenvalues):

$\bar{x}_{1, \dots, 2n}$

Eigenvectors describe the mode shapes:

- 1) relative magnitude of each state
- 2) phase shift between the states

Each component of an eigenvector is real or complex.

$$\bar{x}_{1,2} = \begin{bmatrix} x_1 \\ \vdots \\ x_{2n} \end{bmatrix}$$

$\rightarrow x_1$

$\rightarrow x_{2n}$

Each component is associated with a state.

$$x_i = A \pm \Omega i$$

Mode shape can be visualized on a polar plot:

Example: $s_{1,2} = -1 \pm 1i$ $\bar{x}_{1,2} = [1, 1 \pm 2i, -1 \pm 1i, -2]^T$

